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GRAUERT TUBES AND THE HOMOGENEOUS MONGE-AMPÉRE EQUATION. II

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1. Introduction

Let M be a complex n-dimensional manifold and $\sigma: M \to M$ an anti-holomorphic involution. The fixed point set X of σ is an n-dimensional real-analytic submanifold of M which is "totally real" at all points p (i.e., there exists no nonzero holomorphic vector in $T_pM \otimes \mathbf{C}$ with the property that both its real and its imaginary part are tangent to X). To simplify the exposition below we will also assume that X is compact (though a good deal of what we have to say in the following paragraph is true without this assumption). We recall that the article [8], of which this article is a continuation, has to do with the following well-known theorem of Grauert:

Theorem. *There exists a σ-invariant neighborhood M_1 of X in M and a smooth strictly plurisubharmonic function $\rho: M_1 \to [0, 1)$, such that*

$$X = \rho^{-1}(0) \quad \textit{and} \quad \sigma^* \rho = \rho . \tag{1.1}$$

The main result of [8] is that the function, ρ, in this theorem can be chosen to have an additional property: namely to satisfy the homogeneous Monge-Ampére equation

$$\det\left(\frac{\partial}{\partial z_i \partial \overline{z}_j} \sqrt{\rho}\right) = 0 \tag{1.2}$$

on the compliment of X in M_1. In fact we showed that if X is equipped with a real-analytic Riemannian metric, there exists a unique real analytic solution ρ of (1.2) such that the inclusion of X into M_1 is an isometric imbedding of X (equipped with the given metric) into M_1 equipped with the Kaehler metric

$$\frac{1}{\sqrt{-1}} \sum \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j} d z_1 \, d\overline{z}_j . \tag{$*$}$$

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Our purpose below will be to describe why such ρ's are useful and, in particular, to explain the function-theoretic significance of (1.2).

For the moment let ρ be any function satisfying the hypotheses of Grauert's theorem. The property (1.1) implies that, for all $x \in X$, $d\rho_x = 0$. Moreover, since ρ is strictly plurisubharmonic, the Hessian $d^2\rho_x$ is positive-definite on the normal space to X at x. Therefore, without loss of generality, one can assume that the only critical points of ρ on M_1 are the points of X. Since X is compact, one can also assume without loss of generality that ρ is proper, and, therefore, that the open sets

$$M_\varepsilon = \rho^{-1}([0, \varepsilon^2)), \qquad 0 < \varepsilon < 1, \tag{1.3}$$

form a neighborhood base of X in M. From now on we will refer to these open sets as *Grauert tubes*. Since ρ has no critical points on $M_1 - X$, the boundary of M_ε,

$$\partial M_\varepsilon = \rho^{-1}(\varepsilon^2), \tag{1.4}$$

is smooth, compact, and strictly pseudoconvex, and, as a consequence, M_ε has lots of globally defined holomorphic $(n, 0)$-forms. Let us denote by $\mathscr{O}(\overline{M}_\varepsilon, \Lambda^{n,0})$ the space of holomorphic $(n, 0)$-forms on M_ε which are smooth up to the boundary. We will show in §§2–3 that there is a natural fibration $\pi: M_1 \to X$ which, restricted to $\overline{M}_\varepsilon$, gives rise to a Gysin map:

$$G_\varepsilon: \mathscr{O}(\overline{M}_\varepsilon, \Lambda^{n,0}) \longrightarrow C^\infty(X). \tag{1.5}$$

It has been known for some time [7] that this map is Fredholm; recently Epstein and Melrose [5] have shown that, for sufficiently small ε, it is a bijection. Therefore, given such an ε, there exists, for $0 < \delta < \varepsilon$, a map

$$R_{\varepsilon,\delta}: C^\infty(X) \longrightarrow C^\infty(X) \tag{1.6}$$

defined by the diagram

$$\begin{array}{ccc} C^\infty(X) & \xrightarrow{R_{\varepsilon,\delta}} & C^\infty(X) \\ \uparrow & & \uparrow \\ \mathscr{O}(\overline{M}_\varepsilon, \Lambda^{n,0}) & \longrightarrow & \mathscr{O}(\overline{M}_\delta, \Lambda^{n,0}) \end{array} \tag{1.7}$$

with the vertical arrows the Gysin maps and the bottom arrow the restriction mapping. Notice that, by definition, if $\delta < s < \varepsilon$ then

$$R_{s,\delta} \circ R_{\varepsilon,s} = R_{\varepsilon,\delta}. \tag{1.8}$$

In §4 we will derive from (1.8) an evolution equation of the form

$$\frac{d}{ds}R_{\varepsilon,s} = P_s R_{\varepsilon,s}, \qquad \delta < s < \varepsilon; \tag{1.9}$$

the P_s in this equation is a positive-definite first-order elliptic pseudodifferential operator. We will compute the leading symbol of this operator and show that if ρ satisfies the homogeneous Monge-Ampére equation on the annulus $\delta < \sqrt{\rho} < \varepsilon$, then the symbol of P_s does not depend on s (and conversely). In other words, there exists a positive-definite first-order elliptic pseudodifferential operator P such that

$$P_s = P + Q_s, \tag{1.10}$$

where the Q_s's are *bounded* pseudodifferential operators. We will show, moreover, that if ρ satisfies (1.2) on all of M_ε then

$$P = \sqrt{\Delta}, \tag{1.11}$$

where Δ is the Laplace operator associated with the restriction to X of the Kaehler metric $(*)$.

This result is closely related to a result of Boutet de Monvel on the analytic continuation of the fundamental solution of the wave equation to the imaginary time axis. We will describe the relationship of his result to ours in §6. (We are grateful to Steve Zelditch for calling our attention to this result.)

2. The canonical fibration of the Grauert tube M_ε

The following theorem will play a fundamental role in this paper.

Theorem 2.1. *For every sufficiently small neighborhood U of X in M there exists a unique fibration*

$$\pi: U \longrightarrow X \tag{2.1}$$

such that the one-form

$$\beta = \operatorname{Im} \overline{\partial} \rho \tag{2.2}$$

is π-horizonal at every point p (i.e., if v is tangent to M at p and $d\pi_p(v) = 0$, then $\beta(v) = 0$) and such that, restricted to X, π is the identity map.

This theorem is due to Kostant and Sternberg (see [9, p. 228]). We will give a brief sketch of its proof: Let Ω be the Kaehler form

$$d\beta = \frac{1}{2i}\partial\overline{\partial}\rho, \tag{2.3}$$

and let Ξ be the vector field defined by the identity

$$\iota(\Xi)\Omega = \beta . \tag{2.4}$$

One can show that X is an unstable fixed point set of the one-parameter group of diffeomorphisms generated by Ξ. In fact, there exists a neighborhood U of X with the property that for every $p \in U$ the integral curve $\gamma(t)$, $-\infty < t \leq 0$, of Ξ through p tends to a unique limit point $\pi(p) = \text{Lim}\ \gamma(t)$ as t tends to $-\infty$. Moreover, the mapping

$$\pi : U \longrightarrow X , \qquad p \longrightarrow \pi(p) , \tag{2.5}$$

is a smooth fibration satisfying the hypotheses of the theorem. We will prove now that it is the only such fibration: Suppose π_1 is another such fibration. Then, for every $q \in X$, the restriction of β to the fiber $(\pi_1)^{-1}(q)$ is zero; so $(\pi_1)^{-1}(q)$ is a Lagrangian submanifold of U with respect to the Kaehler form (2.3). Thus, by (2.4), Ξ is tangent to $(\pi_1)^{-1}(q)$. Let p be any point on $(\pi_1)^{-1}(q)$. Then the integral curve of Ξ through p lies on $(\pi_1)^{-1}(q)$, and so the limit to which this curve tends as $t \to -\infty$ is q. Thus $q = \pi_1(p) = \pi(p)$.

Remark. Some additional properties of the vector field Ξ are discussed in the appendix. One property which we will make use of in §4 is that ρ *satisfies the Monge-Ampére equation* (1.2) *if and only if*

$$\Xi\rho = 2\rho . \tag{2.5}$$

For a proof of (2.5) see [8, §5].

Without loss of generality we can assume that U is invariant under the involution σ. Since the fibration $\pi \circ \sigma$ satisfies the hypotheses of Theorem 2.1, the uniqueness of π implies

$$\pi = \pi \circ \delta . \tag{2.6}$$

Next we will give a description of π in terms of the cotangent bundle fibration:

$$\pi_0 : T^*X \longrightarrow X . \tag{2.7}$$

Recall that T^*X is equipped with a canonical one-form $\beta_0 = \Sigma\xi_i\, dx_i$, and with a canonical involution $\sigma_0 : T^*X \to T^*X$, which maps (x, ξ) to $(x, -\xi)$. Moreover, the fixed point set of this involution is just X.

Theorem 2.2. *For every sufficiently small open neighborhood U of X in M there exists a unique open embedding*

$$\Phi : U \longrightarrow T^*X \tag{2.8}$$

such that the restriction of Φ *of* X *is the identity map and such that*

$$\Phi^*\beta_0 = \beta, \tag{2.9}$$

$$\pi = \pi_0 \circ \Phi, \tag{2.10}$$

and

$$\sigma_0 \circ \Phi = \Phi \circ \sigma. \tag{2.11}$$

Proof. To say that the one-form β is π-horizontal is equivalent to saying that for every $p \in U$

$$\beta_p \in \mathrm{Image}(d\pi_p^*).$$

Therefore, one can define a mapping

$$\Phi \colon U \longrightarrow T^*X \tag{2.12}$$

by setting

$$\Phi(p) = (x, \xi) \Leftrightarrow x = \pi(p) \text{ and } (d\pi_p^*)(\xi) = \beta_p. \tag{2.13}$$

It is trivial to check that Φ is the identity on X and that it satisfies (2.9) and (2.10). Moreover, since it satisfies (2.9) it maps the symplectic form $d\beta$ onto the symplectic form $d\beta_0$, and, hence, is locally a diffeomorphism in a neighborhood of every point. Since it is the identity map on X, it is a diffeomorphism on every sufficiently small neighborhood of X in M (and hence, if U is sufficiently small, on U itself). We will leave for the reader to check that Φ is unique. (This amounts to showing that if a diffeomorphism of T^*X onto itself is the identity on X and preserves the canonical one-form $\Sigma\xi_i\, dx_i$, then it is identity; see [1, p. 186].) Finally we note that $\sigma_0 \circ \Phi \circ \sigma$ also satisfies the conditions (2.9) and (2.10); therefore the uniqueness of Φ implies (2.11).

3. The Gysin map

If ε is sufficiently small, then the closure of M_ε is contained in U, and the restriction of π to $\overline{M}_\varepsilon$ is a fibration

$$\pi_\varepsilon \colon \overline{M}_\varepsilon \longrightarrow X \tag{3.1}$$

whose fibers are closed n-balls. Let $C^\infty(\overline{M}_\varepsilon, \Lambda^n)$ be the space of n-forms on M_ε which are smooth up to the boundary. Given an n-form ω belonging to this space, we will associate with it a function $(\pi_\varepsilon)_*\omega$ on X. This function is defined at the point p as the fiber integral

$$\int_{\overline{M}_\varepsilon(p)} \omega \tag{3.2}$$

over the n-ball

$$\overline{M}_\varepsilon(p) = \pi^{-1}(p) \cap \overline{M}_\varepsilon .$$

It is easy to see that this expression depends smoothly on p, and hence defines a linear operator

$$(\pi_\varepsilon)_* : C^\infty(\overline{M}_\varepsilon, \Lambda^n) \longrightarrow C^\infty(X) \tag{3.3}$$

and by restriction a linear operator

$$G_\varepsilon : \mathscr{O}(\overline{M}_\varepsilon, \Lambda^{n,0}) \longrightarrow C^\infty(X) . \tag{3.4}$$

As we mentioned in the introduction, Epstein and Melrose have recently proved

Theorem 3.1 (see [5]). *For ε sufficiently small,* (3.4) *is a bijection.*

G_ε is known to have some very nice microlocal properties (see [3], [5], [7]). We will not attempt to describe these properties in detail; however, roughly speaking, they amount to the following:

Theorem 3.2. *G_ε is an elliptic Fourier integral operator, and its underlying canonical relation is a symplectic mapping*

$$\gamma_\varepsilon : \Sigma_\varepsilon^+ \longrightarrow T^*X - 0 . \tag{3.5}$$

Without getting involved in too may technical details we will try to explain what the various items in this theorem mean: To begin with, ∂M_ε, being the boundary of a strictly pseudo-convex domain, is equipped with a complex of differential operators: the $\overline{\partial}_b$-complex. The characteristic variety of this complex is a conic symplectic submanifold of the contangent bundle of ∂M_ε. It consists of of two connected components, and Σ_ε^+ is what is usually called its "plus" or "inward-pointing" component. Σ_ε^+ can also be described as follows. Let α_ε be the restriction of ∂M_ε of the one-form (2.2). Then

$$\Sigma_\varepsilon^+ = \{(p, c(\alpha_\varepsilon)_p) p \in \partial M_\varepsilon, - c \in \mathbf{R}^+\} . \tag{3.6}$$

Notice that there is a natural identification:

$$\Sigma_\varepsilon^+ = \partial M_\varepsilon \times \mathbf{R}^+ \tag{3.7}$$

identifying $(p, c(\alpha_\varepsilon)_p)$ with (p, c).

Next we will describe the mapping γ_ε. Basically, γ_ε is just the "boundary value" of the Φ occurring in Theorem 2.2. Restricting Φ to ∂M_ε we get an imbedding

$$\Phi_\varepsilon : \partial M_\varepsilon \longrightarrow T^*X - 0 \tag{3.8}$$

which can be extended to a diffeomorphism

$$\gamma_\varepsilon : \partial M_\varepsilon \times \mathbf{R}^+ \longrightarrow T^*X - 0 \tag{3.9}$$

by requiring it to be $\mathbf{R}^+$-equivalent. If we identify Σ_ε^+ with $\partial M_\varepsilon \times \mathbf{R}^+$ (see (3.7)) the mapping (3.9) becomes the mapping (3.5). We will not take the time here to give a precise statement of the first part of theorem; however, the following result explains why the transformation G_ε lives microlocally on Γ_ε.

Theorem 3.3. *Let q be a smooth function on $\overline{M}_\varepsilon$. Then the operator*

$$f \in C^\infty(X) \longrightarrow (\pi_\varepsilon)_* q G_\varepsilon^{-1} f \tag{3.10}$$

is a zeroth order pseudodifferential operator. Moreover, its leading symbol depends only on the restriction of q to ∂M_ε; the pullback of this symbol to ∂M_ε by the mapping (3.8) *is the restriction of q to ∂M_ε.*

Proof. See [4, §11] or [7, §5].

In §4 we will need an analogous result about vector fields, which is also a consequence of the theorems we have just cited: Let $\mathfrak{v}$ be a smooth vector field on $\overline{M}_\varepsilon$, and, for every smooth n-form $\omega \in C^\infty(\overline{M}_\varepsilon, \Lambda^n)$, let $D_\mathfrak{v}\omega$ be the Lie derivative of ω with respect to $\mathfrak{v}$.

Theorem 3.4. *The operator*

$$f \in C^\infty(X) \longrightarrow (\pi_\varepsilon)_* D_\mathfrak{v} G_\varepsilon^{-1} f \tag{3.11}$$

is a first-order pseudodifferential operator, and, moreover, its leading symbol depends only on the restriction of $\mathfrak{v}$ to ∂M_ε.

The explicit receipt for this symbol, which we will describe in a moment, is a little more complicated than in the previous theorem; but it again involves the canonical mapping (3.5). To begin with, suppose that, at the boundary points of M_ε, $\mathfrak{v}$ is tangent to the boundary. Then the restriction of $\mathfrak{v}$ to the boundary is a vector field on the boundary, say $\mathfrak{v}_0$, and, by Lie differentiation, $\mathfrak{v}_0$ defines a first-order differential operator $D_{\mathfrak{v}_0}$ on the space of $(n-1)$-forms on the boundary. Let σ_1 be the restriction of the symbol of this operator to Σ_ε^+ and let σ_2 be the symbol of the operator (3.11). Then

$$\sigma_2 \circ \gamma = \sigma_1 . \tag{3.12}$$

If $\mathfrak{v}$ is *not* tangent to the boundary, one can compute the symbol of (3.11) as follows. Write $\mathfrak{v}$ as a sum

$$\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{w} , \tag{3.13}$$

where $\mathfrak{w}$ is tangent to the boundary of M_ε and $\mathfrak{v}_1$ is of type $(0, 1)$. (In local coordinates,

$$\mathfrak{v}_1 = \sum_{i=1}^{n} f_i \frac{\partial}{\partial \overline{z}_i},$$

where the f_i's are complex valued C^∞ functions.) Then, restricted to $\mathscr{O}(\overline{M}_\varepsilon, \Lambda^{n,0})$, $D_{\mathfrak{v}_1}$ is the multiplication by a C^∞ function; hence, the leading symbol of (3.11) is identical with the leading symbol of the operator

$$f \in C^\infty(X) \longrightarrow (\pi_\varepsilon)_* D_{\mathfrak{w}} G_\varepsilon^{-1} f. \tag{3.14}$$

4. The evolution equation

To derive (1.9) from (1.8) we have to compute the left derivative

$$\left(\frac{d}{ds}\right)_- R_{\varepsilon,s} \tag{4.1}$$

at $s = \varepsilon$. To do this we will first derive a more manageable formula for $R_{\varepsilon,s}$ itself. It is clear from the computations in §2 that

$$\Xi\sqrt{\rho} > 0$$

on $M_\varepsilon - X$. Let f be a smooth function on M_ε which is equal to $-(\Xi\sqrt{\rho})^{-1}$ on the annulus $\delta > \sqrt{\rho} < \varepsilon$, and let

$$\mathfrak{v} = f\,\Xi. \tag{4.2}$$

Then, for $0 < t < \varepsilon - \delta$,

$$\mathfrak{v}\rho = -1,$$

and, therefore, $\exp t\mathfrak{v}$ maps $\overline{M}_\varepsilon$ onto $\overline{M}_{\varepsilon-t}$. Moreover, since Ξ is tangent to the fibers of the fibration π,

$$\pi \circ \exp t\mathfrak{v} = \pi.$$

Hence, for every $p \in X$, $\exp f\mathfrak{v}$ maps the set

$$\overline{M}_\varepsilon(p) = \pi^{-1}(p) \cap \overline{M}_\varepsilon$$

diffeomorphically onto the set

$$\overline{M}_{(\varepsilon-t)}(p) = \pi^{-1}(p) \cap \overline{M}_{\varepsilon-t}.$$

Now let ω be a holomorphic $(n, 0)$-form on M_ε, which is smooth up to the boundary. Then

$$\int_{\overline{M}_{(\varepsilon-t)}(p)} \omega = \int_{\overline{M}_\varepsilon(p)} (\exp t\mathfrak{v})^* \omega . \tag{4.3}$$

As in §2 let $(\pi_\varepsilon)_*$ be the "fiber integration" operator

$$(\pi_\varepsilon)_* \colon C^\infty(\overline{M}_\varepsilon, \Lambda^n) \to C^\infty(X)$$

and G_ε its restriction to $\mathscr{O}(\overline{M}_\varepsilon, \Lambda^{n,0})$. Let f be in $C^\infty(X)$. Applying (4.3) to $\omega = G_\varepsilon^{-1} f$ one obtains:

$$R_{\varepsilon, \varepsilon-t} f = (\pi_\varepsilon)_* (\exp t\mathfrak{v})^* G_\varepsilon^{-1} f ; \tag{4.4}$$

differentiating the left-hand side with respect to t and setting $t = 0$, one obtains for (4.1) the formula

$$\left(\frac{d}{ds}\right)_- R_{\varepsilon, s} f = -(\pi_e)_* D_\mathfrak{v} G_\varepsilon^{-1} f \overset{\text{def}}{=} P_\varepsilon \tag{4.5}$$

at $s = \varepsilon$. By Theorem 3.4, this operator is a pseudodifferential operator of order one, and its leading symbol can be computed by the procedure outlined at the end of §3. Namely, let

$$\mathfrak{v} = \mathfrak{v}_1 + \mathfrak{w} ,$$

where $\mathfrak{v}_1$ is a vector field of type $(0, 1)$ and $\mathfrak{w}$ is a vector field which, at the boundary of M_ε, is tangent to the boundary. Since $\mathfrak{v}$ is tangent to the fibers of π,

$$\langle \mathfrak{v} , \operatorname{Im} \overline{\partial} \rho \rangle = 0 \tag{4.6}$$

or, in other words,

$$\langle \mathfrak{v} , \partial \rho \rangle = \langle \mathfrak{v} , \overline{\partial} \rho \rangle . \tag{4.7}$$

On the other hand,

$$\langle \mathfrak{v} , \overline{\partial} \rho \rangle + \langle \mathfrak{v} , \partial \rho \rangle = \langle \mathfrak{v} , d\rho \rangle = \mathfrak{v}(\sqrt{\rho})^2 = -2\sqrt{\rho} ,$$

so we obtain from (4.7) the identity

$$\langle \mathfrak{v} , \partial \rho \rangle = -\sqrt{\rho} . \tag{4.8}$$

Since $\mathfrak{v}_1$ is of type (0.1), $\langle \mathfrak{v} , d\rho \rangle$ is equal to $\langle \mathfrak{w} , d\rho \rangle$, so, by (4.8),

$$\langle \mathfrak{w} , \partial \rho \rangle = -\sqrt{\rho} . \tag{4.9}$$

Since $\mathfrak{w}$ is tangent to ∂M_ε,

$$\langle \mathfrak{w} , d\rho \rangle = 0 \tag{4.10}$$

on ∂M_ε, from (4.9) we also obtain

$$\langle \mathfrak{w}, \bar\partial\rho\rangle = \sqrt{\rho} \tag{4.11}$$

on ∂M_ε. Recall now that the one-form α_ε is the restriction of $\operatorname{Im}\bar\partial\rho$ to ∂M_ε; so, on ∂M_ε,

$$\langle \mathfrak{w}, \alpha_\varepsilon\rangle = \langle \mathfrak{w}, \operatorname{Im}\bar\partial\rho\rangle = \left\langle \mathfrak{w}, \frac{\bar\partial\rho - \partial\rho}{2i}\right\rangle$$

and, therefore, by (4.10) and (4.11),

$$\langle \mathfrak{w}, \alpha_\varepsilon\rangle = -i\sqrt{\rho} = -i\varepsilon \tag{4.12}$$

on ∂M_ε. Now let $\mathfrak{w}_0$ be the restriction of $\mathfrak{w}$ to ∂M_ε and let σ_1 be, as in (3.12), the symbol of $D_{\mathfrak{w}_0}$, restricted to Σ_ε^+. Then, by (4.12) the value of σ_1 at the point $(p, (\alpha_e)_p) \in \Sigma_\varepsilon^+$ is $-\varepsilon$, so the recipe of §3 gives us the following answer for the symbol of the operator (4.5).

Theorem 4.1. *Let $\Phi: M \to T^*X$ be the mapping Φ in Theorem* 2.2. *Then the symbol of the operator* (4.5) *is equal to $+\varepsilon$ on $\Phi(\partial M_\varepsilon)$.*

Since the symbol of (4.5) is homogeneous for degree one, this theorem determines the symbol on the whole cotangent bundle of X. More explicitly, let ρ_0 be the function $(\Phi^{-1})^*\rho$. Then for every point $(x, \xi) \in T^*X$, there exist a unique $\xi_0 \in T^*X$ and a unique $c \in \mathbf{R}^-$, such that

$$\xi = c\xi_0$$

and $(\sqrt{\rho_0})(x, \xi_0) = \varepsilon$; the theorem states that the symbol of the operator (4.5) at (x, ξ) is $c\varepsilon$. Suppose now that for $a < \varepsilon < b$, this symbol is independent of ε. The computation we have just made shows that a necessary and sufficient condition for this is to be the case is that on the annulus $a < \sqrt{\rho_0} < b$, $\sqrt{\rho_0}$ be a homogeneous function of degree one. In other words, ρ_0 has to satisfy

$$\Xi_0\rho_0 = 2\rho_0,$$

where Ξ_0 is the vector field $\Sigma\xi_i\partial/\partial\xi_i$. However, the diffeomorphism Φ^{-1} maps this vector field onto the vector field Ξ, so this condition is equivalent to

$$\Xi\rho = 2\rho,$$

which is just the Monge-Ampère equation for $\sqrt{\rho}$. Thus, to summarize, we have proved

Theorem 4.2. *The leading symbol of the operator* (4.5) *is independent of ε for ε on the interval $a < \varepsilon < b$ if and only if $\sqrt{\rho}$ satisfies the Monge-Ampère equation* (1.2) *on the annulus $a < \sqrt{\rho} < b$.*

One obtains (1.10) as an immediate corollary of this result. Equality (1.11) follows from this result and a result which we proved in §5 of [8]. Namely, there we showed that if ρ satisfies (1.2) on all of M_ε, then ρ_0 has to be a quadratic function on the cotangent fibers of T^*X and has to be equal to the symbol of $\sqrt{\Delta}$, Δ being the Laplace operator associated with the metric $(*)$.

5. The analytic continuation of the fundamental solution of the wave equation to the imaginary time axis

Let X be equipped with a real-analytic Riemannian metric, and let $P = \sqrt{\Delta}$, where Δ is the Laplace operator associated with this metric. The symbol of P is a real-analytic function on $T^*X - 0$ which is homogeneous of degree one. Let η be the Hamiltonian vector field whose Hamiltonian is this function and let

$$\exp t\eta \colon S^*X \to S^*X$$

be the restriction to the unit cosphere bundle of X of the Hamiltonian flow generated by η. Composing $\exp t\eta$ with the cotangent bundle projection open gets a map

$$\psi_t \colon S^*X \to X,$$

which is real-analytic both with respect to the manifold variables and with respect to t. Therefore one can analytically continue it to the complex t-plane, obtaining, for $\operatorname{Im} t$ sufficiently small, a mapping

$$\psi_t \colon S^*X \to M.$$

For ε sufficiently small, the image of the set

$$\{(x, \xi, t),\ (x, \xi) \in S^*X,\ |\operatorname{Im} t| < \varepsilon\}$$

with respect to the mapping $(x, \xi, t) \to \psi_t(x, \xi)$ is an open subset M_ε of M with a smooth strictly pseudoconvex boundary. Let $\mathscr{O}(\overline{M}_\varepsilon)$ be the ring of homolomorphic functions M_ε which are smooth up to the boundary. The following theorem is due to Boutet de Monvel (see [2]).

Theorem 5.1. *Let* $\exp\sqrt{-1}tP$ *be the one-parameter group of unitary operators generated by* P *and let* $e(x, y, t)$ *be its Schwartz kernel. Then for* ε *sufficiently small:*

1. $e(x, y, t)$ *can be extended to a holomorphic function of* t *on the strip* $0 \le \operatorname{Im} t \le \varepsilon$,
2. *for* y *and* ε *fixed,* $e(x, y, i\varepsilon)$ *can be extended to a holomorphic function of* z, $e(z, y, i\varepsilon)$, *on the tube* M_ε,

3. *for every* C^∞ *function* $f = f(y)$, *the integral*

$$\int e(z, y, i\varepsilon) f(y)\, dy \tag{5.1}$$

is not only holomorphic on M_ε *but is smooth up to the boundary,*

4. *the operator*

$$f \in C^\infty(X) \to \mathscr{O}(\overline{M}_\varepsilon) \tag{5.2}$$

defined by (5.1) *is a Fourier integral operator of complex type,*

5. *for* ε *small enough the operator* (5.2) *is invertible.*

One consequence of Theorem 5.1 is that for every $g \in \mathscr{O}(\overline{M}_\varepsilon)$ there exists an $f \in C^\infty(X)$ such that g is equal to the expression (5.1). However, this means that the restriction of g to X is just $(\exp -\varepsilon P)f$; so from Theorem 5.1 Boutet obtains:

Theorem 5.2. *A necessary and sufficient condition that a function on* X *extend holomorphically to* $\overline{M}_\varepsilon$ *is that it be in the space* $(\exp -\varepsilon P)C^\infty(X)$.

As a corollary of Theorem 5.1 one also gets a simple formula for the restriction operator

$$r_{\varepsilon,\delta}: \mathscr{O}(\overline{M}_\varepsilon) \to (\overline{M}_\delta), \qquad 0 < \delta < \varepsilon. \tag{5.3}$$

Namely, let

$$W_\varepsilon: \mathscr{O}(\overline{M}_\varepsilon) \to C^\infty(X) \tag{5.4}$$

be the inverse of the operator (5.2). Then

$$r_{\varepsilon,\delta} = W_\delta^{-1}(\exp -(\varepsilon - \delta)P)W_\varepsilon, \tag{5.5}$$

which implies that the groupoid of restriction operators $r_{\varepsilon,\delta}$, $0 < \varepsilon > \delta$, is microlocally equivalent to the heat semigroup $\exp -(\varepsilon - \delta)P$, $0 \leq \delta \leq \delta$.

We will briefly describe how Theorems 5.1 and 5.2 are related to the results in §4. To begin with, as we pointed out in the introduction, there is a *unique* real-analytic solution ρ of the Monge-Ampére equation (1.2) having the property that the restriction to X of the Kaehler metric $(*)$ is the given Riemannian metric on X. The main result of our paper [8] states essentially that the Grauert tubes associated with this ρ are the M_ε's in Theorem 5.1.

Let μ be the holomorphic $(n, 0)$-form on M_ε whose restriction to X is the standard Riemannian volume form. For ε sufficiently small every holomorphic $(n, 0)$-form can be written as the product of μ with a holomorphic function, i.e., as $g\mu$, where $g \in \mathscr{O}(\overline{M}_\varepsilon)$.

Theorem 5.3. *There exists an invertible elliptic pseudodifferential operator* K_ε *of order* $\frac{n}{2}$ *such that, for all* $g \in \mathscr{O}(\overline{M}_\varepsilon)$,

$$W_\varepsilon g = (K_\varepsilon G_\varepsilon) g\mu. \tag{5.6}$$

Moreover, K_ε *depends real-analytically on the parameter* ε.

Proof. This is essentially just a restatement of part 4 of Theorem 5.1. q.e.d.

Combining formulas (5.5) and (5.6) one obtains the formula

$$R_{\varepsilon,\delta} = K_\delta^{-1}(\exp -(\varepsilon - \delta)P)K_\varepsilon \tag{5.7}$$

from which one deduces the evolution equation

$$\frac{d}{ds}R_{\varepsilon,s} = P_s R_{\varepsilon,s} \tag{5.8}$$

with

$$P_s = P + Q_s, \tag{5.9}$$

where

$$Q_s = K_s^{-1}([P, K_s] - \dot{K}_s). \tag{5.10}$$

Conversely suppose $R_{\varepsilon,s}$ satisfies an evolution equation of the form (5.6)–(5.7). Then (if all data are real analytic) one can solve the equation

$$\dot{K}_s = [P, K_s] - K_s Q_s \tag{5.10'}$$

for K_s, and, setting $W_s = K_s G_s$, obtain Boutet's formula (5.5).

Appendix A

Let Ξ be the vector field defined by (2.4). We recall that the fibers of the canonical fibration $\pi: M \to X$ are just the *unstable manifolds* of the flow generated by this vector field: Given $p \in M$ let $\gamma(t)$, $-\infty < t < \varepsilon$, be the integral curve of Ξ through p. Then

$$\pi(p) = \operatorname*{Lim}_{t \to -\infty} \gamma(t). \tag{A.1}$$

In this appendix we will describe some other properties of Ξ and in particular give a rather elegant formulation in terms of Ξ of the Monge-Ampére condition (1.2).

a. Here is another description of Ξ: Let η be the Hamiltonian vector field on M associated with the function $\rho/2$. By definition,

$$\imath(\eta)\Omega = \frac{d\rho}{2}. \tag{A.2}$$

Let $J: TM \to TM$ be the vector bundle automorphism defining the complex structure on M. Since Ω is J-invariant,

$$\imath(J\eta)\Omega = -J\frac{d\rho}{2} = \operatorname{Im}\bar{\partial}\rho = \beta,$$

and hence

$$\Xi = J\eta. \tag{A.3}$$

b. Let g be the Kaehler metric $g(\cdot\,,\cdot) = \Omega(J\cdot\,,\cdot)$. Then

$$g(\Xi\,,\cdot) = \Omega(J\Xi\,,\cdot) = -\Omega(\eta\,,\cdot) = -\frac{d\rho}{2}\,,$$

which proves:

Proposition A1. Ξ *is the gradient vector field associated with the function* $-\rho/2$.

c. As we pointed out in §2, the assertion that $\sqrt{\rho}$ satisfies the homogeneous Monge-Ampére equation (1.2) is equivalent to the assertion that

$$\Xi\rho = 2\rho. \tag{A.4}$$

(For the proof of this fact see [8, Proposition 5.1].) Differentiating the identity (2.4) by the vector field η, one gets

$$\imath([\eta\,,\Xi])\Omega = D_\eta\alpha = d(\alpha(\eta)).$$

However,

$$\alpha(\eta) = \Omega(\Xi\,,\eta) = -\Omega(\eta\,,\Xi) = -(\Xi\rho/2)\,,$$

so this shows that

$$\imath([\Xi\,,\eta])\Omega = d\left(\frac{1}{2}\Xi\rho\right). \tag{A.5}$$

On the other hand, by definition

$$\imath(\eta)\Omega = d\left(\frac{1}{2}\rho\right). \tag{A.6}$$

Therefore (A.4) is equivalent to the assertion

$$[\Xi\,,\eta] = \eta. \tag{A.7}$$

Thus we have proved

Proposition A2. *The Monge-Ampére equation* (1.2) *is equivalent to the Lie bracket identity* (A.7).

d. Let $\mu = \operatorname{Log}\rho$ and let τ be the Hamiltonian vector field associated which μ; i.e.,

$$\imath(\tau)\Omega = d\mu. \tag{A.8}$$

We claim that (A.7) is equivalent to

$$[\tau\,,J\tau] = 0. \tag{A.9}$$

Indeed, $\tau = \frac{1}{\rho}\eta$, and $J\tau = \frac{1}{\rho}\Xi$ so

$$\begin{aligned}[J\tau\,,\tau] &= [\rho^{-1}\Xi\,,\rho^{-1}\eta] = \rho^{-2}[\Xi\,,\eta] - \rho^{-3}(\Xi\rho)\eta \\ &= \rho^{-2}(2\,,\eta) - \rho^{-3}(2\rho)\eta = 0.\end{aligned}$$

However, (A.9) implies that the Hamiltonian action of the group of real numbers $\mathbf{R}$ on $M-X$ generated by the Hamiltonian vector field τ extends to a local action of the group of complex numbers $\mathbf{C}$. This action is not, in general, a *holomorphic* action of $\mathbf{C}$, but it has the property that its orbits are complex one-dimensional submanifolds of $M-X$. Thus we have proved:

Theorem. *Let*

$$\mathbf{R} \to \text{Symplecto}\,(M-X) \tag{A.10}$$

be the Hamiltonian action on $\mathbf{R}$ *on* $M-X$ *generated by the Hamiltonian* $\mu = \operatorname{Log} \rho$. *Then* (A.10) *extends to a* (*local*) *action of* $\mathbf{C}$ *on* $M-X$ *if* ρ *satisfies the homogeneous Monge-Ampére equation* (1.2).

Remark. The orbits of this action of $\mathbf{C}$ have another nice characterization: Let $\Omega^{\#}$ be the two-form $(2\mu)^{-1}\partial\bar{\partial}\sqrt{\rho}$. The homogeneous Monge-Ampére equation states that $(\Omega^{\#})^n = 0$; from the fact that ρ is strictly plurisubharmonic it is easy to see that

$$\operatorname{rank} \Omega^{\#}_p = n-1$$

at all points $p \in M-X$. Thus the annihilator of $\Omega^{\#}$ is a two-dimensional integrable subbundle of the tangent bundle of $M-X$. The foliation of $M-X$ which it defines has, as leaves, the orbits of the $\mathbf{C}$-action above. (For more about the structure of these leaves see [4].)

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